# Jordan model for contractions of class $C_{.0}$

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## 1. Preliminaries

a) Let $A=[a_{ik}]$ and $B=[b_{ik}]$ be $n\times m$ matrices over the algebra $H^\infty$ of bounded holomorphic functions in the (open) unit disc, or equivalently, of their (nontangential) limit functions on the unit circle. E. NORDGREN [3] introduced a notion of "quasi-equivalence" for such matrices, which can be defined as follows (*cf.* J. SZŰCS [6]):

Definition. $A$ and $B$ are *quasi-equivalent* if for every (scalar valued) inner function $\omega$ there exist an $n\times n$ matrix $\Phi$ and an $m\times m$ matrix $\Psi$ over $H^\infty$, such that

$$\Phi A = B\Psi, \tag{1.1}$$

$$(\det \Phi)(\det \Psi)\wedge \omega = 1.^{1)} \tag{1.2}$$

Note that this relation is symmetric. Indeed, (1.1) implies

$$\Phi' B = A\Psi' \quad \text{for} \quad \Phi' = (\det \Psi)\Phi^{\wedge} \quad \text{and} \quad \Psi' = (\det \Phi)\Psi^{\wedge}, \tag{1.1}'$$

where the superscript $^\wedge$ denotes algebraic conjugate. As we have

$$\det \Phi' = (\det \Psi)^n (\det \Phi)^{n-1}, \quad \det \Psi' = (\det \Phi)^m (\det \Psi)^{m-1},$$

(1.2) implies

$$(\det \Phi')(\det \Psi')\wedge \omega = 1. \tag{1.2}'$$

It is also obvious that quasi-equivalence is a reflexive and transitive relation.

b) We shall have to do in particular with *inner* functions $A$, $B$, i.e. for which $A^*A=I_m$, $B^*B=I_m$ a.e. on the unit circle. In this case we necessarily have $n\geqq m$.

With every $n\times m$ matrix inner function $C$ we associate an operator[2] $S(C)$

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1) For functions $v_\alpha\in H^\infty$, not all zero, we denote by $\bigwedge_\alpha v_\alpha$ the largest common inner divisor of the $v_\alpha$. In case of a finite system we also use the notation $v_1\wedge\ldots\wedge v_N$ instead of $\bigwedge_1^N v_k$.

on the Hilbert space $\mathfrak{H}(C)$ defined by

$$(1.3)\qquad \mathfrak{H}(C) = H_n^2 \ominus C H_m^2, \quad S(C)u = P_{\mathfrak{H}(C)}(\chi u) \quad \text{for} \quad u \in \mathfrak{H}(C);$$

$H_k^2$ denotes the Hilbert space of (column) $k$-vectors over the space $H^2$ for the unit disc, $P$ denotes orthogonal projection, and

$$\chi(\lambda) = \lambda.$$

c) We shall need among others the following fact from the theory of determinants (see e.g. [5], pp. 26—30):

Let $U$ and $V$ be $n \times m$ matrices over a commutative ring, $n \geqq m$. Then we have

$$(1.4)\qquad \det(V'U) = \sum_{\sigma} \det V_\sigma \cdot \det U_\sigma,$$

where $\sigma$ runs over the set $\Sigma_m^n$ of subsets $\sigma = \{i_1, \ldots, i_m\}$ $(i_1 < i_2 < \cdots < i_m)$ of the set $\{1, \ldots, n\}$, $V'$ denotes the transpose of $V$, and $U_\sigma$, $V_\sigma$ denote the minors of $U$ and $V$ composed of the $m$ rows indicated by $\sigma$. In particular if $U$ is an $n \times m$ matrix $(n \geqq m)$ over the complex field then taking for $V$ the complex conjugate of $U$ we derive from (1.4) that

$$(1.5)\qquad \det(U^* U) = \sum_{\sigma} |\det U_\sigma|^2.$$

This formula readily extends to the case of an $\infty \times m$ matrix $U = [u_{ik}]$ $(i = 1, 2, \ldots;$ $k = 1, \ldots, m)$ with finite $\sum_{i=1}^{\infty} |u_{ik}|^2$ for each $k$ $(k = 1, \ldots, m)$. Thus in particular we have for any isometric matrix $U$, i.e. with $U^* U = I_m$ $(m < \infty)$, the equation

$$(1.6)\qquad 1 = \sum_{\sigma} |\det U_\sigma|^2$$

where $\sigma$ runs over $\Sigma_m^n$ or $\Sigma_m^\infty$ according as the number of rows is a finite number $n (\geqq m)$ or infinite.

d) An operator $X$ from a Hilbert space $\mathfrak{H}$ into a Hilbert space $\mathfrak{H}'$ will be called an *injection* if it is one-to-one, or equivalently, if $\ker X = \{0\}$. A family $\{X_\alpha\}$ of injections $X_\alpha : \mathfrak{H} \to \mathfrak{H}'$ will be called *complete* if

$$\bigvee_{\alpha} X_\alpha \mathfrak{H} = \mathfrak{H}'.$$

Thus this concept of a complete family of injections extends the notion of quasi-affinity for a single operator. Note that if $\{X_\alpha\}$ is a complete family of injections $X_\alpha : \mathfrak{H} \to \mathfrak{H}'$ and $\{X'_\beta\}$ is a complete family of injections $X'_\beta : \mathfrak{H}' \to \mathfrak{H}''$ then $\{X'_\beta X_\alpha\}$ is a complete family of injections $X'_\beta X_\alpha : \mathfrak{H} \to \mathfrak{H}''$.

If $T$ is an operator on $\mathfrak{H}$ and $T'$ is an operator on $\mathfrak{H}'$ we say that $T$ is *injected*

[2]) By operator we mean a continuous linear transformation.

into $T$ if there is an injection $X: \mathfrak{H} \to \mathfrak{H}'$ such that $T'X = XT$; then we write:

$$T' \overset{i}{\succ} T.$$

If there exists even a complete family $\{X_\alpha\}$ of injections $X_\alpha: \mathfrak{H} \to \mathfrak{H}'$ such that $T'X_\alpha = X_\alpha T$ for each $\alpha$ then we write

$$T' \overset{ci}{\succ} T.$$

If this "complete family of injections" can be chosen to consist of a single operator, i.e. if there exists a quasi-affinity $X: \mathfrak{H} \to \mathfrak{H}'$ such that $T'X = XT$, then, according to our earlier adopted terminology, we call $T$ a quasi-affine transform of $T'$ and write

$$T' \succ T.$$

Thus $\succ$ implies $\overset{ci}{\succ}$, and this in turn implies $\overset{i}{\succ}$. Also note that each of these relations is reflexive and transitive. They induce equivalence relations

$$T' \overset{i}{\sim} T, \quad T' \overset{ci}{\sim} T, \quad T' \sim T, \tag{1.7}$$

e.g. $T' \overset{i}{\sim} T$ meaning that both $T' \overset{i}{\succ} T$ and $T \overset{i}{\succ} T'$ hold.

Observe that for operators on finite dimensional space each of these equivalence relations coincides with *similarity*. However, for operators on infinite dimensional space they are different from similarity $T' \approx T$, which requires the existence of a bicontinuous operator $X$ from $\mathfrak{H}$ onto $\mathfrak{H}'$ such that $T'X = XT$.

The equivalence relation $T' \sim T$ was introduced in our previous papers and in our book [H], and called *quasi-similarity*. We shall call the two other equivalence relations in (1.7) *injection-similarity* and *complete injection-similarity,* respectively.

e) For an operator $T$ on $\mathfrak{H}$ the multiplicity $\mu_T$ is defined as the minimal cardinality of a set $\mathfrak{S}$ of vectors in $\mathfrak{H}$ such that $\mathfrak{S}$, $T\mathfrak{S}$, $T^2\mathfrak{S}$, ... span $\mathfrak{H}$. It is immediate that

$$T' \succ T \quad \text{implies} \quad \mu_{T'} \leqq \mu_T. \tag{1.8}$$

More generally, if $T' \overset{ci}{\succ} T$ and if $\mathscr{X} = \{X_\alpha\}$ is a corresponding complete system of injections then

$$\mu_{T'} \leqq (\operatorname{card} \mathscr{X}) \cdot \mu_T. \tag{1.9}$$

Indeed one has only to consider for $T'$ the set $\mathfrak{S}' = \bigcup_\alpha X_\alpha \mathfrak{S}$.

For any contraction $T$ of class $C_{\cdot 0}$ we have

$$\mu_T \leqq \mathfrak{d}_{T^*}; \quad cf. \text{ [8]}. \tag{1.10}$$

For a unilateral shift $S_L$ on $\mathfrak{H}$ the multiplicity $L$ is defined by

$$\dim (\mathfrak{H} \ominus S_L \mathfrak{H}).$$

The two kinds of multiplicity coincide:

(1.11) $$\mu_{S_L} = L.$$

Indeed, if $\mathfrak{S}$ is any "generating" set for $S_L$ then

$$\mathfrak{H} \ominus S_L \mathfrak{H} = \bigvee_{n=0}^{\infty} S_L^n \mathfrak{S} \ominus \bigvee_{n=1}^{\infty} S_L^n \mathfrak{S} = \overline{P[\mathfrak{S}]},$$

where $P$ denotes orthogonal projection onto $\mathfrak{H} \ominus S_L \mathfrak{H}$, and $[\mathfrak{S}]$ denotes the subspace spanned by $\mathfrak{S}$; thus,

$$L = \dim (\mathfrak{H} \ominus S_L \mathfrak{H}) \leqq \dim [\mathfrak{S}] \leqq \operatorname{card} \mathfrak{S},$$

and hence $L \leqq \mu_{S_L}$. Comparing this with (1.10), where in case $T = S_L$ we have $\mathfrak{d}_{T^*} = L$, we get (1.11).

In contrast to (1.11) we have

(1.12) $$\mu_{S_L^*} = 1 \text{ for any (countable) } L \geqq 1:$$

result of D. Sarason, *cf.* [1], Problem 126, or [10].

## 2. Quasi-equivalence of $A$ and $B$ implies complete injection-similarity of $S(A)$ and $S(B)$

We are going to prove the following

Theorem 1. *Let $A$ and $B$ be $n \times m$ matrix valued inner functions over $H^\infty$ ($n \geqq m$) and suppose they are quasi-equivalent. Then $S(A)$ and $S(B)$ are completely injection-similar. Moreover, the corresponding complete systems of injections can be chosen to consist of two injections each, say $\{X, X'\}$ and $\{Y, Y'\}$. If $m = n$, they can be chosen even as singletons $\{X\}$, $\{Y\}$, thus $S(A)$ and $S(B)$ are then quasi-similar.*

Remark. The assertion for the case $n = m$ was already proved in [2].

Proof. As $A$ is inner its values $A(e^{it})$ on the unit circle are isometries, a.e. Thus by (1.6) we have

(2.1) $$1 = \sum_\sigma |\det A_\sigma(e^{it})|^2 \text{ a.e.,}$$

and therefore there exists at least one $\sigma \in \Sigma_m^n$ for which $\det A_\sigma(e^{it})$ is non-zero on a set of positive measure — and therefore a.e. on the unit circle.

Let

$$\omega = \bigwedge_\sigma \det A_\sigma.$$

By virtue of the assumption on $A$ and $B$ to be quasi-equivalent, we readily infer that there exist pairs of matrices, say $\Phi$, $\Psi$ and $\Phi_1$, $\Psi_1$ satisfying (1.1) and such that the conditions

$$(2.2)\qquad (\det \Psi)\wedge\omega = 1, \quad (\det \Psi_1)\wedge\omega = 1,$$

$$(2.3)\qquad (\det \Phi)\wedge(\det \Phi_1) = 1$$

are fulfilled.

In the case $m=n$ it will suffice to choose just one pair, say $\Phi$, $\Psi$, satisfying (1.1) and

$$(2.4)\qquad (\det \Psi)\wedge\omega = 1 \quad \text{and} \quad (\det \Phi)\wedge(\det B) = 1.$$

We first show how (1.1) implies that the operator $X\colon \mathfrak{H}(A)\to\mathfrak{H}(B)$ defined by

$$(2.5)\qquad Xu = P_{\mathfrak{H}(B)}\Phi u \quad \text{for} \quad u\in\mathfrak{H}(A)$$

satisfies the equation

$$(2.6)\qquad S(B)X = XS(A).$$

Indeed, relying on definitions (1.3) and (2.5) we deduce for $u\in\mathfrak{H}(A)$:

$$\begin{aligned} XS(a)u &= P_{\mathfrak{H}(B)}\Phi P_{\mathfrak{H}(A)}(\chi u) = \\ &= P_{\mathfrak{H}(B)}\Phi(\chi u - Aw) \quad \text{for some} \quad w\in H^2_m \\ &= P_{\mathfrak{H}(B)}(\chi\Phi u - B\Psi w) = P_{\mathfrak{H}(B)}(\chi\Phi u) \\ &= P_{\mathfrak{H}(B)}\chi(P_{\mathfrak{H}(B)}\Phi u + Bw') \quad \text{for some} \quad w'\in H^2_m \\ &= P_{\mathfrak{H}(B)}(\chi P_{\mathfrak{H}(B)}\Phi u) = S(B)Xu. \end{aligned}$$

Next we deduce from (the first) condition (2.2) that $X$ is an injection. By virtue of (2.5) we have to show that the condition

$$(2.7)\qquad \Phi u = Bw \quad \text{for some} \quad u\in\mathfrak{H}(A) \quad \text{and} \quad w\in H^2_m$$

implies $u=0$.

To this effect we observe that, by (1.1) and (2.7),

$$\Phi A\Psi^{\wedge}w = B\Psi\Psi^{\wedge}w = (\det \Psi)Bw = (\det \Psi)\,\Phi u.$$

Multiplying on the left by $\Phi^{\wedge}$ and then dividing by $\det\Phi$ we obtain

$$(2.8)\qquad Aw' = (\det \Psi)u, \quad \text{where} \quad w' = \Psi^{\wedge}w\in H^2_m.$$

Now multiply by $A^*$ on the left: as $A$ is an inner function we shall have

$$(2.9)\qquad w' = (\det \Psi)f, \quad \text{where} \quad f = A^*u,$$

a.e. on the unit circle. Note that $f\in L^2_m$.

From (2.8) and (2.9) we also deduce:

$$(\det \Psi)u = Aw' = (\det \Psi)Af;$$

whence,

$$u = Af. \tag{2.10}$$

As we have

$$\Phi u = \begin{cases} Bw & \text{by (2.7)} \\ \Phi Af = B\Psi f & \text{by (2.10) and (1.1),} \end{cases}$$

multiplying by $B^*$ on the left we get

$$w = \Psi f \tag{2.11}$$

a.e. on the unit circle, and therefore

$$(\det \Psi) f = \Psi^{\wedge} \Psi f = \Psi^{\wedge} w \in H^2_m. \tag{2.12}$$

On the other hand, if we denote by $u_\sigma$ and $A_\sigma$ the vector and the matrix formed by the rows of $u$ and $A$ indicated by $\sigma = \{i_1, \ldots, i_m\} \in \Sigma^n_m$, then (2.10) implies $u_\sigma = A_\sigma f$, and therefore

$$(\det A_\sigma) f = A_\sigma^{\wedge} A_\sigma f = A_\sigma^{\wedge} u_\sigma \in H^2_m. \tag{2.13}$$

Now recall condition (2.2): $\Psi$ was chosen so that $\det \Psi$ and $\det A_\sigma$ $(\sigma \in \Sigma^n_m)$ have no non-constant common inner divisor. Thus, applying a lemma of [7] we deduce from (2.12) and (2.13) that

$$f \in H^2_m. \text{ [3]} \tag{2.14}$$

From (2.10) and (2.14) we deduce that $u \in AH^2_m$. Since by assumption we have $u \in \mathfrak{H}(A)$ we conclude that $u = 0$.

Thus we proved that the operator $X$ derived from $\Phi$ by (2.5) is an injection with the intertwining property (2.6). These properties obviously hold for the operator $X_1$ derived from the function $\Phi_1$, as well.

It remains to show that the ranges of $X$ and $X'$ together span the space $\mathfrak{H}(B)$ — and if $m = n$ then so does the range of $X$ alone.

Indeed, we have

$$\begin{aligned} X\mathfrak{H}(A) &= P_{\mathfrak{H}(B)} \Phi \mathfrak{H}(A) = P_{\mathfrak{H}(B)} \Phi [\mathfrak{H}(A) \oplus AH^2_m] \\ &\qquad\qquad \text{because } \Phi AH^2_m = B\Psi H^2_m \subset BH^2_m \perp \mathfrak{H}(B) \\ &= P_{\mathfrak{H}(B)} \Phi H^2_n, \end{aligned}$$

and therefore,

$$X\mathfrak{H}(A) \supset P_{\mathfrak{H}(B)} \Phi (\Phi^{\wedge} H^2_n) = P_{\mathfrak{H}(B)} (\det \Phi) H^2_n,$$

and by the same reason,

$$X_1 \mathfrak{H}(A) \supset P_{\mathfrak{H}(B)} (\det \Phi_1) H^2_n.$$

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[3] This lemma asserts that if $w_\alpha \in H^\infty$ and $w_\alpha f \in H^1$, where $f \in L^1$ and not all $w_\alpha$ are zero, then

$$(\bigwedge_\alpha w_\alpha) f \in H^1.$$

As by condition (2.3)

$$(\det \Phi) \wedge (\det \Phi_1) = 1,$$

we have

$$(\det \Phi) H_n^2 \vee (\det \Phi_1) H_n^2 = H_n^2$$

by Beurling's theorem; hence the ranges of $X$ and $X_1$ span $\mathfrak{H}(B)$.

In the case $m=n$ we have, by the second condition (2.4) and again by Beurling's theorem,

$$(\det \Phi) H_m^2 \vee (\det B) H_m^2 = H_m^2;$$

as $P_{\mathfrak{H}(B)}(\det B) H_m^2 \subset P_{\mathfrak{H}(B)} B H_m^2 = \{0\}$ we conclude that the range of $X$ alone spans $\mathfrak{H}(B)$.

Thus $\{X, X'\}$ is a complete system of injections of $S(A)$ into $S(B)$, and if $m=n$ then $X$ is a quasi-affinity.

The proof of Theorem 1 will be done if only we recall that quasi-equivalence of the matrices $A$ and $B$ is a symmetric relation so that the above constructions can be carried out with the roles of $A$ and $B$ interchanged.

Using inequality (1.8) we deduce from Theorem 1:

Corollary. *For $A$, $B$ as in Theorem* 1, *we have*

$$\mu_{S(A)} \leqq 2\mu_{S(B)}, \quad \mu_{S(B)} \leqq 2\mu_{S(A)}. \tag{2.15}$$

## 3. Jordan model

Now we can refer to a theorem of Nordgren [3] according to which every $n \times m$ matrix $\Theta$ over $H^\infty$ is quasi-equivalent to the corresponding matrix $\Theta'$ in "normal form". If $n \geqq m$ and $\Theta$ has full rank, i.e. has a non-zero minor of order $m$ with non-zero determinant, then $\Theta'$ is defined by

$$\Theta' = \left[\begin{array}{cccc} e_1 & & & 0 \\ & e_2 & & \\ & & \ddots & \\ 0 & & & e_m \\ \hline 0 & 0 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & 0 \end{array}\right] \begin{array}{l} \\ \\ \\ \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} n-m \end{array} \tag{3.1}$$

where $e_1, \dots, e_m$ are the "invariant factors" of $\Theta$. That is,

$$e_k = d_{m-k+1}/d_{m-k} \quad (k = 1, \dots, m), \tag{3.2}$$

where $d_0=1$ and $d_i$ is, for $i=1, \dots, m$, the largest common inner divisor of the determinants of the minors of order $i$; $e_{k+1}$ turns out to be a divisor of $e_k$ ($k=1, \dots, m-1$).

Consider in particular an *inner* function $\Theta$; then $n \geqq m$ and the full rank condition is satisfied, because by (1.6) we have

$$\sum_{\sigma} |\det \Theta_\sigma(e^{it})|^2 = 1 \quad (\sigma \in \Sigma_m^n),$$

a.e. on the unit circle, and therefore there exists at least one $\sigma$ for which $\det \Theta_\sigma \neq 0$.

From (3.1) we deduce that

$$S(\Theta') = S(e_1) \oplus \cdots \oplus S(e_m) \oplus \underbrace{S \oplus \cdots \oplus S}_{n-m}, \tag{3.3}$$

where $S$ is the (simple) unilateral shift

$$S: u \to \lambda u \quad \text{on} \quad H^2.$$

Now it is known (*cf.* [H]) that the general form, up to unitary equivalence, of a contraction $T$ of class $C_{\cdot 0}$[4]) on a (separable) Hilbert space $\mathfrak{H}$, and with *finite* defect indices, say

$$\mathfrak{d}_T = m \quad \text{and} \quad \mathfrak{d}_{T^*} = n,^{5)}$$

is the operator $S(\Theta)$ generated by an $n \times m$ matrix valued, pure,[6]) inner function $\Theta$ over $H^\infty$; $\Theta$ is determined, up to constant unitary factors, uniquely by $T$ (the "characteristic function" of $T$).

Thus, on account of our Theorem 1, every such operator $T$ is completely injection-similar to the corresponding operator $S(\Theta')$ given by (3.3), and if $m=n$ then it is even quasi-similar to $S(\Theta')$.

Generalizing a notation introduced in [9] let us call *Jordan operator* any operator of the form

$$S(p_1) \oplus S(p_2) \oplus \cdots \oplus S(p_R) \oplus \underbrace{S \oplus \cdots \oplus S}_{L \text{ times}}, \tag{3.4}$$

where $p_1, p_2, \dots, p_R$ are non-constant (scalar valued) inner functions, each of which being a divisor of its predecessor, and $R \geqq 0$, $L \geqq 0$; we also use the shorter notation

$$S(p_1, p_2, \dots, p_R) \oplus S_L. \tag{3.4$'$}$$

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[4]) For the definition of the classes $C_{\cdot 0}$, $C_{10}$, etc. *cf.* [H].

[5]) $\mathfrak{d}_T = \dim \overline{D_T \mathfrak{H}}$ where $D_T = (I - T^* T)^{1/2}$.

[6]) I.e., $\|\Theta(0)a\| < \|a\|$ for every non-zero, constant $m$-vector $a$.

Note that this is a contraction of class $C_{\cdot 0}$ with defect indices $R$ and $R+L$, so that $L$ is the difference of the defect indices. Also note that since $S(1)$ is the trivial operator on the space $\mathfrak{H}(1)=\{0\}$, we can omit from the sum (3.3) the terms $S(e_k)$ (if any) for which $e_k=1$ and obtain in such a way a Jordan operator with non-constant inner functions $e_k$.

We have therefore the following:

Theorem 2. *Every contraction $T$ of class $C_{\cdot 0}$ on a separable Hilbert space, with finite defect indices, say $\mathfrak{d}_T=m$ and $\mathfrak{d}_{T^*}=n$ ($n\geqq m$), is completely injection-similar (and if $n=m$ even quasi-similar) to the Jordan operator*

$$(3.5)\qquad J = S(e_1, e_2, \ldots, e_K)\oplus S_{n-m}$$

*formed by those invariant factors $e_k$ of the characteristic function $\Theta$ of $T$, which are non-constant; we have*

$$(3.6)\qquad \mu_T \leqq 2\mu_J, \quad \mu_J \leqq 2\mu_T.$$

Remark. If $n=m$ quasi-similarity of $T$ to a Jordan-operator was first proved in [9]; another proof, exhibiting the functions $e_k$ as the invariant factors of the characteristic function was given, in case $n=m$, in [2].

Now we turn to prove that even *uniqueness* holds.

Theorem 3. *The only Jordan operator injection-similar to $T$ is the canonical one given by* (3.5).

On account of Theorem 2, Theorem 3 will be established if we prove:

Theorem 4. *Let*

$$J = S(q_1, \ldots, q_r)\oplus S_l \quad \textit{and} \quad J' = S(p_1, \ldots, p_R)\oplus S_L$$

*be Jordan operators. $J$ can be injected into $J'$ if and only if $l\leqq L$, $r\leqq R$, and each $q_k$ is a divisor of $p_k$ ($k=1, \ldots, r$).*

Proof. That the conditions are sufficient, is obvious (use the fact that $S_l$ and $S(q)$ are unitarily equivalent to parts of $S_L$ and $S(p)$, respectively, in invariant subspaces, whenever $l\leqq L$ and $q$ is an inner divisor of $p$).

To prove necessity first observe that an injection $X$ of $J$ into $J'$ induces an injection of $S_l$ into $J'$. Now as $J'$ is of class $C_{\cdot 0}$ and has defect indices $\mathfrak{d}_{J'}=R$ and $\mathfrak{d}_{J'^*}=R+L$, inequality $l\leqq L$ is a consequence of Theorem 5, to be proved in Sec. 4.

Next observe that $X$ also induces an injection $X_0$ of $S(Q)=S(q_1, \ldots, q_r)$ into $J'$. Since

$$J'^k X_0 = X_0 S(Q)^k \to 0 \quad \text{as} \quad k \to \infty$$

this implies that the range of $X_0$ is contained in the space of the $C_{00}$ part of $J'$, i.e. we can consider $X_0$ as an injection of $S(Q)$ into $S(P)=S(p_1, \dots, p_R)$:

$$X_0\colon \mathfrak{H}(Q) \to \mathfrak{H}(P), \quad S(P)X_0 = X_0 S(Q). \tag{3.7}$$

Hence we infer that $r \leqq R$ and that $q_k$ is a divisor of $p_k$ $(k=1, \dots, r)$ by the same arguments as in [9], namely in the following way.

We begin by considering an arbitrary inner function $w$ and define

$$\mathfrak{M} = \overline{w(S(Q))\mathfrak{H}(Q)}, \quad M = S(Q)|\mathfrak{M},$$

$$\mathfrak{M}' = \overline{w(S(P))\mathfrak{H}(P)}, \quad M' = S(P)|\mathfrak{M}',$$

and $Y = X_0|\mathfrak{M}$. Clearly, $M\colon \mathfrak{M} \to \mathfrak{M}$, $M'\colon \mathfrak{M}' \to \mathfrak{M}'$, and as (3.7) implies

$$w(S(P))X_0 = X_0 w(S(Q))$$

we have

$$Y\colon \mathfrak{M} \to \mathfrak{M}'.$$

Since $X_0$ is injective so is $Y$. From (3.7) it also follows

$$M'Y = S(P)X_0|\mathfrak{M} = X_0 S(Q)|\mathfrak{M} = YM,$$

i.e. $M$ can be injected into $M'$.

Next observe that

$$\mathfrak{M} = \bigoplus_{i=1}^{r} \mathfrak{M}_i, \quad \text{where} \quad \mathfrak{M}_i = \overline{w(S(q_i))\mathfrak{H}(q_i)},$$

$$\mathfrak{M}' = \bigoplus_{j=1}^{R} \mathfrak{M}'_j, \quad \text{where} \quad \mathfrak{M}'_j = \overline{w(S(p_j))\mathfrak{H}(p_j)}$$

and accordingly,

$$M = \bigoplus_{i=1}^{r} M_i, \quad \text{where} \quad M_i = S(q_i)|\mathfrak{M}_i,$$

$$M' = \bigoplus_{j=1}^{R} M'_j, \quad \text{where} \quad M'_j = S(p_j)|\mathfrak{M}'_j.$$

Now $M_i$ is unitarily equivalent to $S(q_i^w)$ and $M'_j$ is unitarily equivalent to $S(p_j^w)$, where

$$q_i^w = \frac{q_i}{q_i \wedge w}, \quad p_j^w = \frac{p_j}{p_j \wedge w}.$$ [7]

[7]) This fact was used, but not quite explicitly explained in [9]. An explicit treatment follows at the end of this section.

[By virtue of Lemma 2 of [9], in each one of the sequences

$$q_1^w, q_2^w, \dots, q_r^w \quad \text{and} \quad p_1^w, p_2^w, \dots, p_R^w,$$

each term is a divisor of its predecessor.] Then $M$ and $M'$ are unitarily equivalent respectively to

$$M^w = \bigoplus_{i=1}^{r} S(q_i^w) \quad \text{and} \quad M'^w = \bigoplus_{j=1}^{R} S(p_j^w),$$

and hence $M^w$ can also be injected into $M'^w$.

Now we apply Proposition 2 of [9] to $M^w$ and $M'^w$ remarking that in the proof on pp. 103—104 of [9] the injective property of $X$ is only used. We infer that the number of non-constant $q^w$ cannot exceed the number of non-constant $p^w$.

In particular, taking $w=1$ this gives $r \leqq R$.

Take now $w=p_k$ for a fixed $k \leqq r$. As $\frac{p_j}{p_j \wedge p_k}=1$ for $j \geqq k$, the number of non-constant $p^w$ is, in this case, less than $k$. Therefore, $q_k^w$ must be constant, i.e. we have $\frac{q_k}{q_k \wedge p_k}=1$, thus $q_k$ is divisor of $p_k$.

This concludes the proof of Theorem 4.

*

For sake of completeness we are going to make explicit the *unitary equivalence,* for any inner functions $q$ and $w$, of the operators

$$S\left(\frac{q}{q \wedge w}\right) \quad \text{and} \quad S(q)|\mathfrak{A}, \quad \text{where} \quad \mathfrak{A} = \overline{w(S(q))\mathfrak{H}(q)}.$$

First, observe that

$$w(S(q))\mathfrak{H}(q) = P_{\mathfrak{H}(q)}(w \cdot \mathfrak{H}(q)) = P_{\mathfrak{H}(q)}(wH^2 + qH^2),$$

and therefore[8])

$$\mathfrak{A} = \overline{P_{\mathfrak{H}(q)}((w \wedge q)H^2)} = q[\bar{q}(w \wedge q)H^2]_- ,^{9})$$

where $[\cdot]_-$ denotes orthogonal projection from $L^2$ onto $L^2 \ominus H^2$. Hence,

$$\overline{q \wedge w} \cdot \mathfrak{A} = \frac{q}{q \wedge w}\left[\overline{\left(\frac{q}{q \wedge w}\right)} H^2\right]_- = \mathfrak{H}\left(\frac{q}{q \wedge w}\right),$$

and we infer that multiplication by $\overline{q \wedge w}$ is a unitary operator $W$ from $\mathfrak{A}$ onto $\mathfrak{H}\left(\frac{q}{q \wedge w}\right)$.

8) A superscript bar denotes closure or complex conjugate, depending on the context.

9) It is easy to prove that

$$P_{\mathfrak{H}(q)} u = q[\bar{q}u]_- \quad \text{for any} \quad u \in L^2.$$

This implies in pacticular that $[\bar{q}H^2]_-$ equals $\bar{q}\mathfrak{H}(q)$, and hence is closed.

For any $h \in \mathfrak{A}$ we have, on the unit circle,

$$S\left(\frac{q}{q\wedge w}\right) Wh = S\left(\frac{q}{q\wedge w}\right)(\overline{q\wedge wh}) = \frac{q}{q\wedge w}\left[\overline{\left(\frac{q}{q\wedge w}\right)\cdot \chi \cdot \overline{q\wedge w}\cdot h}\right]_{-} =$$

$$= \frac{q}{q\wedge w}[\bar{q}\chi h]_{-} = \overline{q\wedge w}\cdot S(q)h = WS(q)h,$$

and therefore,

$$S\left(\frac{q}{q\wedge w}\right) W = W(S(q)|\mathfrak{A}).$$

## 4. Injection of unilateral shifts. Shift index of an operator

**1.** We are going to prove a statement which we already referred to in the proof of Theorem 4.

Theorem 5. *Let $T$ be a contraction of class $C_{\cdot 0}$ with finite defect indices $\mathfrak{d}_T = m$ and $\mathfrak{d}_{T^*} = n$ ($m \leqq n$). If a unilateral shift $S_\alpha$, of multiplicity $\alpha$, can be injected into $T$ then $\alpha \leqq n - m$.*

Proof. Suppose $S_\alpha$ can be injected into $T$ for some $\alpha > n - m$. Then $S_{n-m+1}$ can also be injected into $T$; thus considering the model $S(\Theta)$ of $T$ we have $S(\Theta)X = XS_{n-m+1}$ for $\Theta = \Theta_T$ and an injection

$$X: H^2_{n-m+1} \to \mathfrak{H}(\Theta);$$

$\Theta$ is an $n \times m$ matrix valued, pure inner function. By the Lifting Theorem (cf. [H], Theorem VI.3.6) we have

$$Xu = P_{\mathfrak{H}(\Theta)} \Xi u \quad \text{for} \quad u \in H^2_{n-m+1}, \tag{4.1}$$

where $\Xi$ is some $n \times (n-m+1)$ matrix over $H^\infty$. Obviously, we have $Xu = 0$ for some $u \in H^2_{n-m+1}$ if

$$\Xi u = \Theta w \quad \text{for some} \quad w \in H^2_m. \tag{4.2}$$

Consider (4.2) as a linear system of equations in the $(n-m+1)+m = n+1$ unknowns $u_1, \ldots, u_{n-m+1}, w_1, \ldots, w_m$. Since this system consists of $n$ equations, there exists a non-zero solution $[u, w]$ in the quotient field derived from the algebra $H^\infty$; multiplying by the smallest common multiple of the denominators we get a non-zero solution $[u, w]$ over $H^\infty$. Then $u$ must be also non-zero; otherwise (4.2) would imply $\Theta w = 0$, $w = \Theta^* \Theta w = 0$, thus $[u, w]$ would be zero. Thus $u \neq 0$ so that $X$ is not an injection: a contraction which achieves the proof.

Theorem 5 has the following complement:

Theorem 6. *If $T$ is a contraction of class $C_{\cdot 0}$ with finite defect index $\mathfrak{d}_T=m$, then $S_{n-m}$ can be injected in $T$ whether or not $\mathfrak{d}_{T^*}=n$ is finite or infinite.*

Proof. Considering $T$ in its model $S(\Theta)$ we wish to find an injection $X$: $H^2_\alpha \to \mathfrak{H}(\Theta)$ satisfying $S(\Theta)X=XS_\alpha$, where $\alpha=n-m$. By virtue of the same Lifting Theorem as above this means to find an $n\times\alpha$ matrix $\Xi$ over $H^\infty$ such that condition

$$\Xi u = \Theta w \quad \text{for some} \quad u\in H^2_\alpha \quad \text{and} \quad w\in H^2_m \tag{4.3}$$

implies $\mu=0$.

As a consequence of formula (1.6) there exists $\sigma\in\Sigma^n_m$ such that $\Theta_\sigma$ has non-zero determinant. Choose for $\Xi$ the (constant) matrix such that

$\Xi_\sigma=0$ (the zero $m\times\alpha$ matrix), $\Xi_{\bar\sigma}=I_\alpha$ (the $\alpha\times\alpha$ unit matrix),

where $\bar\sigma$ denotes the complement of $\sigma$ in $\{1, \dots, n\}$ or $\{1, 2, \dots\}$ according as $n<\infty$ or $n=\infty$. Then we have $\Xi_\sigma u=0$ so (4.3) implies $\Theta_\sigma w=0$, and hence $w=0$. Therefore,

$$u = I_\alpha u = \Xi_{\bar\sigma} u = \Theta_{\bar\sigma} w = 0.$$

Thus, $X$ is an injection and the proof is done.

Combination of Theorems 5 and 6 gives:

Theorem 5/6. *If $T$ is a contraction of class $C_{\cdot 0}$, with $\mathfrak{d}_T=m<\infty$, then $S_\alpha$ can be injected into $T$ if and only if $\alpha\leqq n-m$, where $n=\mathfrak{d}_{T^*}(\leqq\infty)$.*

**2.** Let us define, for any operator $T$, the number

$$\varkappa_T = \sup\{\alpha: S_\alpha \text{ can be injected into } T\} \tag{4.4}$$

and call it the *shift index* of $T$; $\varkappa_T$ is a non-negative integer or $\infty(=\aleph_0)$.

Theorem 5/6 expresses that for a contraction $T$ of class $C_{\cdot 0}$, with finite defect index $\mathfrak{d}_T$, we have

$$\varkappa_T = \mathfrak{d}_{T^*} - \mathfrak{d}_T, \tag{4.5}$$

and this supremum is *attained* even if $\mathfrak{d}_{T^*}=\infty$.

On the other hand, if $T$ is *any contraction of class* $C_0$ (i.e. completely non-unitary and such that $\varphi(T)=0$ for some inner function $\varphi$) then

$$\varkappa_T = 0.$$

Indeed, if $TX=XS$ for some injection $X$ then we also have $\varphi(T)X=X\varphi(S)$ and therefore $\varphi(S)=0$. But this is impossible since $\varphi(S)$ is an isometry: restriction of the unitary operator $\varphi(U)$, where $U$ is the (simple) bilateral shift extending the unilateral shift $S$.

Further examples were studied in [10]: For every *non-algebraic strict contraction* $T$ we have $\varkappa_T=\infty$. Also, we have $\varkappa_{S^*}=\infty$, and in both cases the value $\infty$ is actually *attained* in (4.4).

From the definition (4.4) of $\varkappa_T$ we immediately infer the inequality

$$\varkappa_T \geqq \sum_j \varkappa_{T_j} \quad \text{if} \quad T \overset{i}{\succ} \bigoplus_j T_j. \tag{4.6}$$

and in particular

$$\varkappa_T \geqq \varkappa_{T'} \quad \text{if } T' \text{ is a restriction of } T \text{ to an invariant subspace.} \tag{4.7}$$

As an application consider the case of a $T \in C_{\cdot 0}$ with finite $\mathfrak{d}_T$. Then $T' \in C_{\cdot 0}$, and $I'-T'^*T'$ is a restriction of $P'(I-T^*T)$; hence $\mathfrak{d}_{T'} \leqq \mathfrak{d}_T$; thus by (4.7) and (4.5)

$$\mathfrak{d}_{T^*}-\mathfrak{d}_T \geqq \mathfrak{d}_{T'^*}-\mathfrak{d}_{T'}, \quad \mathfrak{d}_{T'^*} \leqq \mathfrak{d}_{T^*}-(\mathfrak{d}_T-\mathfrak{d}_{T'})$$

and therefore,

$$\mathfrak{d}_T = \mathfrak{d}_{T'}+p \quad \text{and} \quad \mathfrak{d}_{T^*} \geqq \mathfrak{d}_{T'^*}+p \quad \text{with some} \quad p \geqq 0. \tag{4.8}$$

Let us note that we can arrive at the results (4.8) also by applying the connections between invariant parts of $T$ and regular factorizations of its characteristic functions. Also note that for completely non-unitary contractions of general type we have by [H], Proposition VII.3.6,

$$\mathfrak{d}_{T'} \geqq \mathfrak{d}_T \quad \text{and} \quad \mathfrak{d}_{T'^*} \leqq \mathfrak{d}_{T^*}+\mathfrak{d}_T.$$

**3.** For another application of Theorem 5 consider an operator $T$ such that

$$T \overset{i}{\prec} S_k \quad \text{for some} \quad k \geqq 1 \quad \text{(finite or infinite).} \tag{4.9}$$

Then there exists an injection $X$ such that $S_kX=XT$. The closure of the range of $X$ is invariant for $S_k$; let $S_k'$ be the restriction of $S_k$ to this invariant subspace: $S_k'$ is also a unilateral shift and its multiplicity $h$ is $\leqq k$. (Consequence of the analogous fact for bilateral shifts; [H], Proposition I.2.1.) As we have $S_k' \succ T$ it follows that $S_h \succ T$. From the relations

$$S_h \succ T \overset{i}{\succ} S_k$$

we infer $S_h \overset{i}{\succ} S_k$. Since $S_h \in C_{10}$ and $\varkappa_{S_h}=h$, Theorem 5 implies that $h \geqq k$. Thus $h=k$, and hence

$$S_k \succ T, \quad S_k^* \prec T^*.$$

Recalling (1.8) and (1.10—12) we obtain

$$\mu_T \geqq \mu_{S_k}=k, \quad \mu_{T^*} \leqq \mu_{S_k^*}=1.$$

Thus we have proved:

Proposition 1. *For any operator $T$ satisfying condition* (4.9) *we have*

$$(4.10)\qquad S_k \succ T, \quad \mu_T \geqq k, \quad \mu_{T^*} = 1.$$

Corollary 1. *If both* (4.9) *and* $T \succ S_k$ *hold then*

$$T \sim S_k, \quad \mu_T = k.$$

Observe that if $T$ is a *contraction* of class $C_{10}$ with finite defect indices then its Jordan model (3.5) cannot contain a non-zero $C_{0\cdot}$ part since otherwise $T$ also contained a non-zero $C_{0\cdot}$ part; therefore the model reduces to the unilateral shift part, i.e. we have

$$T \overset{ci}{\sim} S_k \quad \text{with} \quad k = \varkappa_T = \mathfrak{d}_{T^*} - \mathfrak{d}_T.$$

(It is obvious that, conversely, $S_k \overset{i}{\succ} T$ for some $k$ implies $T \in C_{1\cdot}$.) So Proposition 1 has:

Corollary 2. *For every contraction $T$ of class $C_{10}$ with finite defect indices we have $S_k \succ T \overset{ci}{\succ} S_k$, where $k = \mathfrak{d}_{T^*} - \mathfrak{d}_T$, and $\mu_T \geqq k$, $\mu_{T^*} = 1$.*

## 5. An example

As an illuminating example we are going to study in detail the contractions $T$ of class $C_{10}$, with defect indices $\mathfrak{d}_T = 1$ and $\mathfrak{d}_{T^*} = 2$, or equivalently, the operators $T = S(\Theta)$ associated with purely contractive inner and $*$-outer functions of the form

$$\Theta = \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix},$$

i.e. for which

$$(5.1)\qquad \vartheta_1, \vartheta_2 \in H^\infty, \quad |\vartheta_1(0)| < 1, \quad |\vartheta_2(0)| < 1,$$

$$(5.2)\qquad |\vartheta_1(e^{it})|^2 + |\vartheta_2(e^{it})|^2 = 1$$

a.e. on the unit circle, and

$$(5.3)\qquad \vartheta_1^\sim H^2 + \vartheta_2^\sim H^2 \text{ is dense in } H^2.$$ [10])

By a theorem of Beurling condition (5.3) is equivalent to the condition $\vartheta_1^\sim \wedge \vartheta_2^\sim = 1$ and this in turn is equivalent to

$$(5.3)'\qquad \vartheta_1 \wedge \vartheta_2 = 1.$$

[10]) $u^\sim(\lambda) = \overline{u(\bar\lambda)}$ for scalar valued, and $A^\sim(\lambda) = A(\bar\lambda)^*$ for operator valued functions.

By Corollary 2 to Proposition 1 we have then

(5.4) $$S \succ T \overset{cl}{\succ} S \quad \text{and} \quad \mu_T \geqq 1, \quad \mu_{T^*} = 1.$$

The question arises whether we have even $T \succ S$ and, as a consequence, $T \sim S$?

To this effect let us try to find a quasi-affinity $X: H^2 \to \mathfrak{H}(\Theta)$ such that

$$S(\Theta)X = XS.$$

By virtue of the Lifting Theorem ([H], Theorem VI.3.6) the operators $X$ satisfying this equation are precisely those which result in the form

(5.5) $$Xu = P_{\mathfrak{H}(\Theta)} \Xi u \quad (u \in H^2)$$

from some "matrix" $\Xi = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ over $H^\infty$.

On account of (5.5) the range of $X$ is dense in $\mathfrak{H}(\Theta)$ if and only if

(5.6) $$\Xi H^2 + \Theta H^2 \text{ is dense in } H_2^2.$$

As $\vartheta_1 \wedge \vartheta_2 = 1$ implies, by a theorem of Beurling, that $[\vartheta_2, -\vartheta_1] H_2^2$ is dense in $H^2$, and as $[\vartheta_2, -\vartheta_1]\Xi = \vartheta_2 x_1 - \vartheta_1 x_2$ and $[\vartheta_2, -\vartheta_1]\Theta = 0$, condition (5.6) implies that

(5.7) $$(\vartheta_2 x_1 - \vartheta_1 x_2) H^2 \text{ is dense in } H^2,$$

which, again by a theorem of Beurling, means that

(5.7)′ $$\vartheta_2 x_1 - \vartheta_1 x_2 \text{ is an } \textit{outer} \text{ function.}$$

Conversely, (5.7)′ implies (5.6) because

$$\Xi H^2 + \Theta H^2 = [\Xi, \Theta] H_2^2 \supset [\Xi, \Theta][\Xi, \Theta]^\wedge H_2^2 = (\det[\Xi, \Theta]) H_2^2 = (\vartheta_2 x_1 - \vartheta_1 x_2) H_2^2.$$

For $\Xi$ satisfying (5.7)′ the operator $X$ has also the property of being an injection. On account of (5.5) we have to show to this effect that if $\Xi u = \Theta w$ for some $u, w \in H^2$ then $u = 0$. Now our assumption can also be written in the form $[\Xi, \Theta] \begin{bmatrix} u \\ -w \end{bmatrix} = 0$; whence $(\det[\Xi, \Theta]) \begin{bmatrix} u \\ -w \end{bmatrix} = [\Xi, \Theta]^\wedge \cdot [\Xi, \Theta] \begin{bmatrix} u \\ -w \end{bmatrix} = 0$. As $\det[\Xi, \Theta]$ is an outer function and therefore is not zero this implies $u = 0$.

Thus we have proved so far that $T \sim S$ *if and only if*

(5.8) $$\vartheta_2 x_1 - \vartheta_1 x_2 \text{ is outer for some } x_1, x_2 \in H^\infty.$$

Let us find an operator theoretic meaning of condition (5.8).

We know that (5.8) implies $T \sim S$, and hence $\mu_T = 1$. Let us show that, conversely, $\mu_T = 1$ implies (5.8).

Thus suppose that $T$ $(=S(\Theta))$ has a cyclic vector $\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix}$, i.e. the vectors $S(\Theta)^n\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix}$ $(n=0, 1, \dots)$ span $\mathfrak{H}(\Theta)$. Then the set

$$\left\{\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix} e^{int}+\Theta H^2\right\}_{n=0}^{\infty} \quad \text{spans} \quad H_2^2$$

and therefore (multiplying on the left by the $1\times 2$ matrix valued outer function $[\vartheta_2, -\vartheta_1]$) the set

$$\{(\vartheta_2\xi_1-\vartheta_1\xi_2)e^{int}\}_{n=0}^{\infty} \quad \text{spans} \quad H^2,$$

thus $\vartheta_2\xi_1-\vartheta_1\xi_2$ is a (scalar valued) outer function. Note that $\xi_1$, $\xi_2$ are in $H^2$ but not necessarily in $H^\infty$. We can construct $x_1, x_2 \in H^\infty$ such that $\vartheta_2 x_1-\vartheta_1 x_2$ is also outer, in the following way. The function

$$g(t) = [|\xi_1(e^{it})|^2+|\xi_2(e^{it})|^2+1]^{-1/2}$$

obviously satisfies

$$0 \leqq g(t) \leqq 1 \quad \text{and} \quad |\log g(t)| \leqq \frac{1}{2}[|\xi_1(e^{it})|^2+|\xi_2(e^{it})|^2] \in L^1.$$

Hence we infer that the outer function

$$h(\lambda) = \exp\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+\lambda}{e^{it}-\lambda}\log g(t)\,dt \quad (|\lambda|<1)$$

belongs to $H^\infty$ and satisfies $|h(e^{it})|=g(t)$ a.e. on the unit circle. Then $x_1=\xi_1 h$ and $x_2=\xi_2 h$ belong to $H^2$, and moreover, since

$$|x_k(e^{it})| = |\xi_k(e^{it})|g(t) \leqq 1 \quad (k = 1, 2),$$

we conclude that $x_1$, $x_2$ actually belong to $H^\infty$, and we have

$$\vartheta_2 x_1-\vartheta_1 x_2 = (\vartheta_2\xi_1-\vartheta_1\xi_2)h = \text{outer}\times\text{outer} = \text{outer}.$$

Summing up, we have proved:

Proposition 2. *For a contraction $T=S(\Theta)$ of class $C_{10}$, with defect indices $\mathfrak{d}_T=1$ and $\mathfrak{d}_{T^*}=2$, the following conditions are equivalent:*

(i) $T\sim S$,
(ii) $T\succ S$,
(iii) $\mu_T=1$,
(iv) $\vartheta_2 x_1-\vartheta_1 x_2$ *is an outer function for some* $x_1, x_2\in H^\infty$.

Now there do exist functions $\Theta=\begin{bmatrix}\vartheta_1\\ \vartheta_2\end{bmatrix}$ for which (5.1)—(5.3)′ hold, but (5.8) does not. Such is the case when

$$\vartheta_1 = \frac{1}{\sqrt{2}}B, \quad \vartheta_2 = \frac{1}{\sqrt{2}}E,$$

where $B$ is an infinite Blaschke product with zeros $a_n = 1 - b_n$ $(0 \leqq b_n < 1,\ \sum' b_n < \infty)$ and $E$ is the "singular" inner function

$$A(\lambda) = \exp\frac{\lambda+1}{\lambda-1}.$$

The property $B \wedge E = 1$ is obvious. For the fact that $Bx + Ey$ will not be outer at any choice of $x, y \in H^\infty$ (noticed by the second author at an early stage of the present investigations), *see* NORDGREN [4].

Thus for the operator $T = S(\Theta)$ corresponding to this example we have $\mu_T > 1$. As on other hand by (3.6) $\mu_T \leqq 2\mu_S = 2$ it follows that $\mu_T = 2$. So we have proved:

Proposition 3. *The Jordan model $J$ of an operator $T$ of the type considered in Theorem 2 is completely injection-similar, but not always quasi-similar to $T$. While $\mu_T \leqq 2\mu_J$ always holds it occurs that $\mu_T \neq \mu_J$ and even that $\mu_T = 2\mu_J$.*

Thus injection-similarity, and even complete injection-similarity, are definitely *weaker* relations than quasi-similarity.

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